

KUMMER SURFACES ASSOCIATED TO (1,2)-POLARIZED ABELIAN SURFACES

AFSANEH MEHRAN

ABSTRACT. The aim of this paper is to describe the geometry of the generic Kummer surface associated to a $(1,2)$ -polarized abelian surface. We show that it is the double cover of a weak del Pezzo surface and that it inherits from the del Pezzo surface an elliptic fibration with twelve singular fibers of type I_2 .

1. INTRODUCTION

The extensive study of Kummer surfaces is explained by their rich geometry and their multiple roles in the theory of K3 surfaces and beyond [Hud90, PŠŠ71, Bea85].

Let A be an abelian surface and consider the involution which maps a to $-a$ for any a in A . This involution has sixteen fixed points, namely the sixteen two-torsion points of A . The quotient surface has sixteen ordinary double points and its minimal resolution is a smooth K3 surface called the Kummer surface associated to A and denoted by $\text{Kum}(A)$. Nikulin proved that any K3 surface containing sixteen disjoint smooth rational curves is a Kummer surface [Nik75].

Given a Kummer surface $\text{Kum}(A)$, there is a natural way of constructing new Kummer surfaces from it. One takes the minimal model of the double cover of $\text{Kum}(A)$ branched along eight disjoint smooth rational curves C_1, \dots, C_8 , that are even (see section 2) and that are orthogonal in $\text{Pic}(\text{Kum}(A))$ to eight other smooth rational curves. We obtain in this way a new Kummer surface $\text{Kum}(B)$ together with a rational map $\text{Kum}(B) \xrightarrow{\tau} \text{Kum}(A)$.

In the second section of the paper, we explain this construction in details and show that the abelian surface associated to the new Kummer surface $\text{Kum}(B)$ is isogenous to A . In fact we prove that the map τ is induced by an isogeny of degree two on the associated abelian surfaces.

In section 3, we describe the geometry of a generic jacobian Kummer surface and explain its classical double plane model. We also recall a theorem of Naruki [Nar91] giving explicit generators of the Néron-Severi lattice of a generic jacobian Kummer surface.

In section 4, we apply the construction of section 2 to the generic jacobian Kummer surface. We obtain in this way, fifteen non isomorphic Kummer surfaces which are associated to $(1,2)$ -polarized abelian surfaces.

Finally in section 5, we show that the Kummer surfaces of section 4 admit an elliptic fibration with twelve singular fibers of the type I_2 . We also prove that these Kummer

surfaces are double cover of a weak Del Pezzo surface (i.e. the blowup of \mathbb{P}^2 at seven points) and that for each of our Kummer surfaces there exists a decomposition of a very degenerate sextic \mathcal{S} (see figure 1) into a quartic Q and a conic C for which we have the theorem

Theorem 1.1. *The rational double cover $\text{Kum}(B) \xrightarrow{-\tau} \text{Kum}(A)$ decomposes as*

$$\begin{array}{ccc} \text{Kum}(B) & \xrightarrow{\varphi} & T \\ \downarrow \tau & & \downarrow \zeta \\ \text{Kum}(A) & \xrightarrow{\phi} & \mathbb{P}^2 \end{array}$$

where ϕ is the canonical resolution of the double cover of \mathbb{P}^2 branched along \mathcal{S} . The maps ζ and φ are the canonical resolutions of the double covers branched along Q and $\zeta^*(C)$ respectively.

2. EVEN EIGHT AND KUMMER SURFACE

We will now introduce the notion of an even eight and the double cover construction associated to it. By applying this construction to special even eights of a Kummer surface, we obtain new Kummer surfaces.

Definition 2.1. Let Y be a K3 surface, an *even eight* on Y is a set of eight disjoint smooth rational curves C_1, \dots, C_8 , for which $C_1 + \dots + C_8 \in 2S_Y$. Here S_Y denotes the Néron-Severi group of Y .

If C_1, \dots, C_8 , is an even eight on a K3 surface Y , then there is a double cover $Z \xrightarrow{p} Y$, branched on $C_1 + \dots + C_8$. If E_i denotes the inverse image of C_i , then $p^*(C_i) = 2E_i$ and $E_i^2 = -1$. Hence, we may blowdown the E_i 's to the surface X and obtain the diagram

$$\begin{array}{ccc} Z & \xrightarrow{\epsilon} & X \\ p \downarrow & \nearrow 2:1 & \\ Y & & \end{array}$$

It turns out that the surface X is again a K3 surface and the covering involution $\iota : X \rightarrow X$ is symplectic with eight fixed points [Nik75].

Suppose now that the K3 surface Y is a Kummer surface, we want to exhibit natural even eights lying on it. For this purpose, we recall a central lemma of Nikulin.

Lemma 2.2. [Nik75] *Let Y be a Kummer surface and let $E_1, \dots, E_{16} \subset Y$ be sixteen smooth disjoint rational curves. Denote by $I = \{1, \dots, 16\}$ the set of indices for the curves E_i 's and by $Q = \{M \subset I \mid \frac{1}{2} \sum_{i \in M} E_i \in S_Y\}$; then for every M in Q , we have $\#M = 8$ or 16 and there exists on I a unique 4-dimensional affine geometry structure over \mathbb{F}_2 , whose hyperplanes consist of the subsets $M \in Q$ containing eight elements.*

The existence of such a 4-dimensional affine geometry implies that $I \in Q$ or equivalently that $\sum_{i=1}^{16} E_i \in 2S_Y$. We can proceed exactly as for an even eight and take the double cover $V \xrightarrow{p} Y$ branched along $E_1 + \cdots + E_{16}$. Again we blowdown the preimage of the E_i 's to a surface A and obtain the diagram

$$\begin{array}{ccc} V & \xrightarrow{\epsilon} & A \\ p \downarrow & \nearrow \pi_A & \\ Y & & \end{array}$$

The difference with this diagram and the one above is that now the surface A is an abelian surface and that the map π_A realizes Y as the Kummer surface associated to A . We point out that by uniqueness, the affine geometry on I corresponds to the one existing on A_2 , the set of 2-torsion points on A . [Nik75].

It follows also from the lemma that there exist on $Y (\simeq \text{Kum}(A))$ thirty even eights, denoted by M_1, \dots, M_{30} , i.e. the thirty affine hyperplanes of I .

Let $M \in \{M_1, \dots, M_{30}\}$ be one of these even eights. We can assume that M consists of the curves E_1, \dots, E_8 . The curves E_9, \dots, E_{16} are then orthogonal to M , i.e.

$$E_i \cdot E_j = 0 \text{ if } 1 \leq i \leq 8 \text{ and } 9 \leq j \leq 16.$$

If $X \xrightarrow{\tau} \text{Kum}(A)$ is the double cover associated to M , then the K3 surface X contains again sixteen disjoint smooth rational curves. Indeed since the curves E_9, \dots, E_{16} do not intersect the branch locus of the double cover $p : Z \rightarrow Y$, they split under p and define sixteen disjoint smooth rational curves on Z . These sixteen curves are then isomorphically mapped by the blowdown $Z \xrightarrow{\epsilon} X$ to sixteen curves on X . It follows that X contains sixteen disjoint smooth rational curves and hence it is a Kummer surface.

Proposition 2.3. *Let M be an even eight on a Kummer surface $\text{Kum}(A)$ such as above, then the K3 surface X associated to M is a Kummer surface. Moreover there is an abelian surface B associated to X for which we have the commutative diagram*

$$\begin{array}{ccc} B & \xrightarrow{p} & A \\ \pi_B \downarrow & & \pi_A \downarrow \\ X = \text{Kum}(B) & \xrightarrow{\tau} & \text{Kum}(A) \end{array}$$

where $B \xrightarrow{p} A$ is an isogeny of degree two.

Proof. Since we have already shown that X is a Kummer surface, we only have to prove that B is degree two isogenous to A . Write the abelian surface A as the complex torus \mathbb{C}^2/Λ and let $E_9, \dots, E_{16} \subset \text{Kum}(A)$ be the eight disjoint smooth rational curves orthogonal to M . These curves also form an even eight and hence they correspond to an affine hyperplane H in A_2 . Up to translation we can fix the origin on A in H . Let $\frac{[v]}{2}$ be the generator of A_2/H , it defines a sublattice $\Lambda' \subset \Lambda$. Explicitly we have that

$\Lambda' = \mathbb{Z}h_1 \oplus \mathbb{Z}h_2 \oplus \mathbb{Z}h_3 \oplus \mathbb{Z}2v$, where $H = \langle \frac{[h_1]}{2}, \frac{[h_2]}{2}, \frac{[h_3]}{2} \rangle \subset A_2$. The canonical inclusion $\Lambda' \hookrightarrow \Lambda$, induces the following commutative diagram :

$$\begin{array}{ccc} \mathbb{C}^2/\Lambda' & \xrightarrow{p} & \mathbb{C}^2/\Lambda \\ \pi' \downarrow & & \downarrow \pi \\ \text{Kum}(\mathbb{C}^2/\Lambda') & \xrightarrow{q} & \text{Kum}(\mathbb{C}^2/\Lambda) \end{array}$$

where p is an isogeny of degree two. The covering involution of p is given by the translation by the 2-torsion point $[v]$ in \mathbb{C}^2/Λ' . It induces the symplectic involution on $\text{Kum}(\mathbb{C}^2/\Lambda')$

$$\sigma : \text{Kum}(\mathbb{C}^2/\Lambda') \rightarrow \text{Kum}(\mathbb{C}^2/\Lambda')$$

which has exactly eight fixed points [Nik79], namely the projection of the sixteen points on \mathbb{C}^2/Λ' satisfying

$$[z] + [v] = -[z], \text{ or equivalently } 2[z] = [v].$$

The isogeny p maps the set $\{[z] \in \mathbb{C}^2/\Lambda' \mid 2[z] = [v]\}$ to $A_2 - H$. In other words, the affine hyperplane $A_2 - H$ corresponds to the even eight M in $\text{Kum}(\mathbb{C}^2/\Lambda)$. Hence the resolution of the rational map q is exactly the double cover of $\text{Kum}(A)$ branched along M and the abelian surface \mathbb{C}/Λ' is B .

□

3. JACOBIAN KUMMER SURFACE

In this section we briefly expose the classical geometry of a jacobian Kummer surface and its beautiful 16_6 -configuration. We describe its double plane model and give explicit generators for its Néron-Severi lattice. This description follows a paper of Naruki [Nar91].

A Kummer surface $\text{Kum}(A)$ is said to be a jacobian Kummer surface if the surface A is the jacobian of a curve C of genus two. Moreover, it is a generic jacobian Kummer surface if its Picard rank is 17.

Recall that the degree two map given by the linear system $|2C|$, $A \xrightarrow{|2C|} \mathbb{P}^3$, factors through the involution $a \mapsto -a$, and hence defines an embedding $A/\{1, i\} \hookrightarrow \mathbb{P}^3$. The image of this map is a quartic $Y_0 \subset \mathbb{P}^3$ with sixteen nodes. Denote by L_0 the class of a hyperplane section of Y_0 . Projecting Y_0 from a node defines a rational map $Y_0 \xrightarrow{2:1} \mathbb{P}^2$. We blowup the center of projection

$$\begin{array}{ccc} Y_1 & & \\ \downarrow & \searrow & \\ Y_0 & \dashrightarrow & \mathbb{P}^2 \end{array}$$

and we call $E_1 \subset Y_1$ the exceptional divisor and $L_1 \subset Y_1$ the pullback of a line in \mathbb{P}^2 . Finally we resolve the remaining fifteen singularities of Y_1 and obtain the Kummer surface

$\text{Kum}(A)$ and a map of degree two $\text{Kum}(A) \xrightarrow{\phi} \mathbb{P}^2$. The map ϕ is given by the linear system $|L - E_0|$, where L and E_0 are the pullback of L_1 and E_1 respectively.

The branch locus of the map ϕ is a reducible plane sextic \mathcal{S} , which is the union of six lines, l_1, \dots, l_6 , all tangent to a conic W .

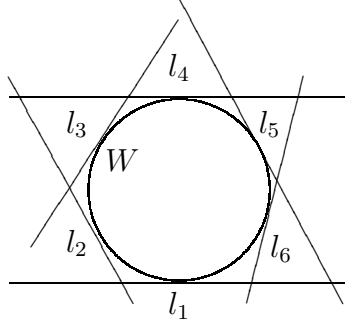


FIGURE 1. The sextic \mathcal{S}

Let $p_{ij} = l_i \cap l_j \in \mathbb{P}^2$, where $1 \leq i < j \leq 6$. Index the ten $(3,3)$ -partitions of the set $\{1, 2, \dots, 6\}$, by the pair (i, j) with $2 \leq i < j \leq 6$. Each pair (i, j) defines a plane conic l_{ij} passing through the sextuplet $p_{1i}, p_{1j}, p_{ij}, p_{lm}, p_{ln}, p_{mn}$, where $\{l, m, n\}$ is the complement of $\{1, i, j\}$ in $\{1, 2, \dots, 6\}$ and where $l < m < n$. The map ϕ factors as

$$\text{Kum}(A) \xrightarrow{\tilde{\phi}} \tilde{\mathbb{P}}^2 \xrightarrow{\eta} \mathbb{P}^2$$

where η is the blowup of \mathbb{P}^2 at the p_{ij} 's and where $\tilde{\phi}$ is the double cover of $\tilde{\mathbb{P}}^2$ branched along the strict transform of the plane sextic \mathcal{S} in $\tilde{\mathbb{P}}^2$. Denote by $E_{ij} \subset \text{Kum}(A)$ the preimage of the exceptional curves of $\tilde{\mathbb{P}}^2$. The ramification of the map $\tilde{\phi}$ consists of the union of six disjoint smooth rational curves $C_0 + C_{12} + C_{13} + C_{14} + C_{15} + C_{16}$. The preimage of the ten plane conics l_{ij} defines ten more smooth disjoint rational curves $C_{ij} \subset \text{Kum}(A)$, $2 \leq i < j \leq 6$. Finally, note that $\phi(E_0) = W$. The sixteen curves E_0, E_{ij} $2 \leq i < j \leq 6$ are called the *nodes* of $\text{Kum}(A)$ and the sixteen curves C_0, C_{ij} , $2 \leq i < j \leq 6$ are called the *tropes* of $\text{Kum}(A)$. These two sets of smooth rational curves satisfy a beautiful configuration called the 16_6 -configuration, i.e. each node intersects exactly six tropes and vice versa.

It is now possible to fully describe the Néron-Severi lattice $S_{\text{Kum}(A)}$ of a general jacobian Kummer surface.

Theorem 3.1. [Nar91] *Let $\text{Kum}(A)$ be a generic jacobian Kummer surface. Its Néron-Severi lattice $S_{\text{Kum}(A)}$ is generated by the classes of E_0, E_{ij}, C_0, C_{ij} and L , with the relations:*

- (1) $C_0 = \frac{1}{2}(L - E_0 - \sum_{i=2}^6 E_{1i})$,
- (2) $C_{1j} = \frac{1}{2}(L - E_0 - E_{1j} - \dots - E_{j-1j} - E_{jj+1} - \dots - E_{j6})$, where $2 \leq j \leq 6$,

- (3) $C_{jk} = \frac{1}{2}(L - E_{1j} - E_{1k} - E_{jk} - E_{lm} - E_{ln} - E_{mn})$ where $2 \leq i < j \leq 6$, and $\{l, m, n\}$ are as described above.

The intersection pairing is given by:

- (1) the E_0, E_{ij} are mutually orthogonal,
- (2) $\langle L, L \rangle = 4, \langle L, E_0 \rangle = \langle L, E_{ij} \rangle = 0$,
- (3) $\langle E_0, E_0 \rangle = \langle E_{ij}, E_{ij} \rangle = -2$,
- (4) the C_0, C_{ij} are mutually orthogonal,
- (5) $\langle L, C_0 \rangle = \langle L, C_{ij} \rangle = 2$.

The action on $S_{\text{Kum}(A)}$ of the covering involution α of the map ϕ is given by:

$$\begin{aligned} \alpha(C_0) &= C_0 & \alpha(C_{1j}) &= C_{1j} & 2 \leq j \leq 6 \\ \alpha(E_{ij}) &= E_{ij} & \alpha(L) &= 3L - 4E_0 \\ \alpha(E_0) &= 2L - 3E_0 & \alpha(C_{ij}) &= C_{ij} + L - 2E_0 & 2 \leq i < j \leq 6. \end{aligned}$$

Remark 3.2. The minimal resolution of the double cover of \mathbb{P}^2 branched along the sextic \mathcal{S} in figure 1 is a Kummer surface (see [Hud90] for a proof).

4. (1,2)-POLARIZED KUMMER SURFACES

In this section, we apply the construction of section 2 to a generic jacobian Kummer surface. We identify all the even eights made out of its nodes and study the associated Kummer surfaces. First we recall some standard facts about the polarization of abelian varieties.

A polarization on a complex torus \mathbb{C}^g/Λ is the class of an ample line bundle L in its the Néron-Severi group. As the latter group is equal, for abelian varieties, to the group of hermitian forms H on \mathbb{C}^g , satisfying $E = \text{Im}H(\Lambda, \Lambda) \subset \mathbb{Z}$, the ample line bundle L corresponds to a positive definite hermitian forms E_L . According to the elementary divisor theorem, there exists a basis $\lambda_1, \dots, \lambda_g, \mu_1, \dots, \mu_g$ of Λ , with respect to which E_L is given by the matrix

$$\begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix} \text{ with } D = \begin{pmatrix} d_1 & 0 & 0 & \dots \\ 0 & d_2 & 0 & \dots \\ \vdots & 0 & \ddots & 0 \\ \vdots & \vdots & 0 & d_g \end{pmatrix}$$

where $d_i \geq 0$ and $d_i | d_{i+1}$ for $i = 1, \dots, g-1$. The vector (d_1, d_2, \dots, d_g) is the type of the line bundle L .

Example 4.1. [BL04]

- (1) If $J(C)$ is the Jacobian of a curve C of genus two, then the line bundle associated to the divisor C is a polarization of type $(1, 1)$.
- (2) If L is a polarization of type (d_1, \dots, d_g) on a complex torus, then $\chi(L) = d_1 \cdots d_g$.

- (3) If $X_1 \xrightarrow{p} X_2$ is an isogney of degree 2 of abelian surfaces and L is a polarization of type $(1, 1)$ on X_2 , then $\chi(p^*(L)) = 2\chi(L) = 2 \cdot 1$. Hence $p^*(L)$ is a polarization of type $(1, 2)$ on X_1 .

Proposition 4.2. *Let $\text{Kum}(A)$ be a generic jacobian Kummer surface and let E_0, E_{ij} , $1 \leq i < j \leq 6$ be its sixteen nodes. There exist fifteen even eights made out of its nodes that do not contain E_0 . These even eights are of the form*

$$\Delta_{i,j} = E_{1i} + \cdots + \widehat{E}_{ij} + \cdots + E_{i6} + E_{1j} + \cdots + \widehat{E}_{ij} + \cdots + E_{j6},$$

where $1 \leq i < j \leq 6$ and $E_{11} = 0$.

The Kummer surface $\text{Kum}(B_{ij})$ obtained from the double cover branched along Δ_{ij} is associated to an abelian surface B_{ij} with a $(1, 2)$ -polarization.

Proof. For any couple (i, j) with $1 \leq i < j \leq 6$, consider the divisor $2C_{1i} + 2C_{1j}$, where we set $C_{11} := C_0$. According to the description of the Néron-Severi lattice of a general jacobian Kummer surface in the previous section, we have the equality

$$2C_{1i} + 2C_{1j} = 2(L - E_0) - (E_{1i} + \cdots + E_{ij} + \cdots + E_{i6} + E_{1j} + \cdots + E_{ij} + \cdots + E_{j6}).$$

Therefore

$$2C_{1i} + 2C_{1j} - 2(L - E_0) + 2E_{ij} = E_{1i} + \cdots + \widehat{E}_{ij} + \cdots + E_{i6} + E_{1j} + \cdots + \widehat{E}_{ij} + \cdots + E_{j6}$$

and consequently

$$E_{1i} + \cdots + \widehat{E}_{ij} + \cdots + E_{i6} + E_{1j} + \cdots + \widehat{E}_{ij} + \cdots + E_{j6}$$

is an even eight not containing E_0 . As there are exactly fifteen choices for i and j , we obtain this way all of the possible even eight.

Let $\text{Kum}(B_{ij})$ be the Kummer surface obtained by taking the double cover branched along such an even eight. By the proposition 2.3, the surface B_{ij} is degree two isogenous to A . Since A has a $(1, 1)$ -polarization, it follows from the example 4.1 that B_{ij} has a $(1, 2)$ -polarization. \square

The reason why we only consider the even eights not containing E_0 is because we would obtain the exact same Kummer surface whether we take the double cover branched along an even eight or its complement (see proof of the proposition 2.3).

In the remaining of this section, we will prove that no two of the Kummer surfaces $\text{Kum}(B_{ij})$ are isomorphic.

Definition 4.3. The *Nikulin lattice* is an even lattice N of rank eight generated by $\{c_i\}_{i=1}^8$ and $d = \frac{1}{2} \sum_{i=1}^8 c_i$, with the bilinear form $c_i \cdot c_j = -2\delta_{ij}$.

Remark 4.4. If C_1, \dots, C_8 is an even eight on a K3 surface, then the primitive sublattice generated by the class of the C_i 's in the Néron-Severi group of X is a Nikulin lattice.

The following proposition gives a condition on two even eights to give rise to non isomorphic K3 surfaces.

Proposition 4.5. *Let Y be a Kummer surface and let Δ_1 and Δ_2 be two even eights on Y . Denote by X_1 and X_2 the respective double covers of Y . If $N_1, N_2 \subset S_Y$ are the two Nikulin lattices corresponding to Δ_1 and Δ_2 , then*

$$X_1 \simeq X_2 \iff \exists f \in \text{Aut}(Y) \text{ such that } f^*(N_1) = N_2.$$

Proof. We suppose that X_1 is isomorphic to X_2 and we denote by $X_2 \xrightarrow{g} X_1$ an isomorphism between X_2 and X_1 . Let $X_1 \xrightarrow{i_1} Y$ and $X_2 \xrightarrow{i_2} Y$ be the covering involutions with respect to the rational double covers $X_1 \dashrightarrow Y$ and $X_2 \dashrightarrow Y$.

Claim: The following diagram is commutative:

$$\begin{array}{ccc} H^2(X_1, \mathbb{Z}) & \xrightarrow{g^*} & H^2(X_2, \mathbb{Z}) \\ \downarrow i_1^* & & \downarrow i_2^* \\ H^2(X_1, \mathbb{Z}) & \xrightarrow{g^*} & H^2(X_2, \mathbb{Z}). \end{array}$$

Proof of the claim: Suppose that the above diagram does not commute. Then the surface X_1 would admit two distinct symplectic involutions, namely i_1 and $g \circ i_2 \circ g^{-1}$. Moreover the quotient of X_1 by both of these involutions would be birational to the same Kummer surface Y . In [Meh07], it is shown that the rational double cover of a Kummer surface $\text{Kum}(A)$ is determined by an embedding $T_X \hookrightarrow T_A$ preserving the Hodge decomposition of T_X and T_A . Since there is a unique embedding of T_X into T_A which preserves the Hodge decomposition, it follows that $i_1 = g^{-1} \circ i_2 \circ g$.

Hence $i_2 \circ g = g \circ i_1$ and the isomorphism g descends to an isomorphism on the quotients

$$X_2/i_2 \xrightarrow{g} X_1/i_1.$$

Since this isomorphism maps the eight singular points of X_2/i_2 to the eight singular points of X_1/i_1 , it extends to an automorphism $Y \xrightarrow{f} Y$, for which $f^*(N_1) = N_2$.

Conversely, let $Y \xrightarrow{f} Y$ be an automorphism of Y for which $f^*(N_1) = N_2$. Denote by $Z_i \xrightarrow{p_i} Y$ the double cover of Y branched along the even eight N_i for $i = 1, 2$. Consider the fiber product

$$\begin{array}{ccc} Z_1 \times_Y Y & \xrightarrow{q} & Z_1 \\ \downarrow p & & \downarrow p_1 \\ Y & \xrightarrow{f} & Y. \end{array}$$

The map $Z_1 \times_Y Y \xrightarrow{p} Y$ is a double cover of Y branched along the even set N_2 or equivalently $Z_1 \times_Y Y = Z_2$. Similarly, by considering the fiber product

$$\begin{array}{ccc} Z_2 \times_Y Y & \xrightarrow{h} & Z_2 \\ \downarrow r & & \downarrow p_2 \\ Y & \xrightarrow{f} & Y, \end{array}$$

we see that $Z_2 \times_Y Y = Z_1$. The maps h and $q = h^{-1}$ define an isomorphism between Z_1 and Z_2 which induces the required isomorphism between X_1 and X_2 . \square

Using the same notation as in the proposition 4.2, we prove the following theorem

Proposition 4.6. *Let Δ_{ij} and $\Delta_{i'j'}$ be two even eights defined as in proposition 4.2.*

$$\text{Kum}(B_{ij}) \simeq \text{Kum}(B_{i'j'}) \Leftrightarrow \{i, j\} = \{i', j'\}.$$

Proof. It is clear that if $\{i, j\} = \{i', j'\}$, then the corresponding Kummer surfaces are equal. Thus we only have to prove the other direction. Without loss of generality, we may assume that $\Delta_{i'j'} = \Delta_{12}$ and we suppose that there exists f an automorphism of $\text{Kum}(A)$ for which $f^*(\Delta_{12}) = \Delta_{ij}$.

Claim:

$$\{f^*(E_{13}), f^*(E_{14}), f^*(E_{15}), f^*(E_{16}), f^*(E_{23}), f^*(E_{24}), f^*(E_{25}), f^*(E_{26})\} = \\ \{E_{1i}, \dots, \hat{E}_{ij}, \dots, E_{i6}, E_{1j}, \dots, \hat{E}_{ij}, \dots, E_{j6}\}$$

Proof of the claim: Let N be a Nikulin lattice and let $D \in N$ be a divisor represented by a smooth rational curve. Note that since D is an effective reduced divisor and N is negative definite, then $D^2 = -2$. It is therefore sufficient to show that the only -2 -classes in N are the c_i 's and the claim will follow. We write D as $D = \sum_{i=1}^8 \lambda_i c_i + \epsilon d$ where $\lambda_j \in \mathbb{Z}$ and $\epsilon = 0$ or 1 . If $\epsilon = 1$, then the equality

$$D^2 = -2 \sum_{i=1}^8 \lambda_i^2 - 2 \sum_{i=1}^8 \lambda_i - 4 = -2$$

implies that $\sum_{i=1}^8 \lambda_i^2 + \lambda_i = -1$. Since the latter equation has no integer solution, we conclude that $\epsilon = 0$. Hence

$$D^2 = -2 \sum_{i=1}^8 \lambda_i^2 = -2$$

or equivalently, $\sum_{i=1}^8 \lambda_i^2 = 1$. Therefore there exists a unique λ_k for which $\lambda_k = 1$ and $\lambda_i = 0$ for $i \neq k$.

In [Keu97], it is proven that any automorphism of a jacobian generic Kummer surface induces \pm identity on $D_{S_{\text{Kum}(A)}}$ where $D_{S_{\text{Kum}(A)}}$ is the discriminant group $S_{\text{Kum}(A)}^*/S_{\text{Kum}(A)}$. We want to apply this fact to the automorphism f . We consider the action of f^* on the following two independent elements of $D_{S_{\text{Kum}(A)}}$

$$\frac{1}{2}(E_{13} + E_{14} + E_{23} + E_{24}) \text{ and } \frac{1}{2}(E_{12} + E_{23} + E_{15} + E_{35}).$$

From the claim, we deduce that

$$f^*(E_{13} + E_{14} + E_{23} + E_{24}) = E_{i_1 i} + E_{i_2 i} + E_{j_1 j} + E_{j_2 j}$$

for some classes $E_{i_1 i}, E_{i_2 i}, E_{j_1 j}, E_{j_2 j} \in \Delta_{ij}$.

From the identity $f_{D_{S_{\text{Kum}(A)}}}^* = \pm \text{id}_{D_{S_{\text{Kum}(A)}}}$, we also deduce that

$$f^*\left(\frac{1}{2}(E_{13} + E_{14} + E_{23} + E_{24})\right) = \pm \frac{1}{2}(E_{13} + E_{14} + E_{23} + E_{24}).$$

Putting these two informations together we find that

$$E_{13} + E_{14} + E_{23} + E_{24} + E_{i_1 i} + E_{i_2 i} + E_{j_1 j} + E_{j_2 j} \in 2S_Y.$$

Since the only even eights containing $E_{13}, E_{14}, E_{23}, E_{24}$ are Δ_{12} and Δ_{34} , we deduce that $\Delta_{ij} = \Delta_{34}$. We proceed similarly for $f^*(E_{12} + E_{23} + E_{13} + E_{35})$ and find that Δ_{ij} must be equal to Δ_{25} which yields to a contradiction. \square

Corollary 4.7. *The fifteen Kummer surfaces $\text{Kum}(B_{ij})$ are not isomorphic.*

5. ELLIPTIC FIBRATION AND WEAK DEL PEZZO SURFACE

In this section, we provide an alternate description of the Kummer surfaces $\text{Kum}(B_{ij})$ as the double covers of a weak del Pezzo surface. We relate this construction to the projective double plane model of the generic jacobian Kummer surface of section 3. First we note the existence on $\text{Kum}(B_{ij})$ of an elliptic fibration that will be useful later. For simplicity, we will always argue for the Kummer surface $\text{Kum}(B_{12})$.

Proposition 5.1. *Let $\text{Kum}(B_{12})$ be the Kummer surface constructed in the proposition 4.2. The surface $\text{Kum}(B_{12})$ admits a Weierstrass elliptic fibration with exactly twelve singular fibers of the type I_2 .*

Proof. Let $\text{Kum}(A) \xrightarrow{\phi} \mathbb{P}^2$ be the double plane model of the generic jacobian Kummer surface introduced in section 3. Consider the pencil of lines passing through the point p_{12} in \mathbb{P}^2 . Its preimage in $\text{Kum}(A)$ defines an elliptic fibration, given by the divisor class $F = L - E_0 - E_{12}$. The divisors

$$F_1 = E_{15} + E_{16} + 2C_0 + E_{13} + E_{14}, \quad \text{and} \quad F_2 = E_{25} + E_{26} + 2C_{12} + E_{23} + E_{24}$$

define two fibers of type I_0^* of this fibration. Moreover, the six divisors

$$\begin{aligned} F_3 &= L - E_0 - E_{12} - E_{45} + E_{45}, \\ F_4 &= L - E_0 - E_{12} - E_{46} + E_{46}, \\ F_5 &= L - E_0 - E_{12} - E_{35} + E_{35}, \\ F_6 &= L - E_0 - E_{12} - E_{36} + E_{36}, \\ F_7 &= L - E_0 - E_{12} - E_{34} + E_{34}, \\ F_8 &= L - E_0 - E_{12} - E_{56} + E_{56} \end{aligned}$$

define six I_2 fibers. Since the Euler characteristics of the F_i 's add up to 24, which is equal to the Euler characteristic of a K3 surface, we conclude by Shioda's formula [Shi90] that the F_i 's are the only singular fibers of the elliptic fibration defined by the linear system $|F|$. Note also that the curves C_{13}, C_{14}, C_{15} and C_{16} are sections of this fibration.

We now analyze the induced fibration τ^*F on $\text{Kum}(B_{12})$, where $\text{Kum}(B_{12}) \dashrightarrow \text{Kum}(A)$ is the rational double cover defined by the even eight Δ_{12} . We remark that the even eight Δ_{12} satisfies

$$\Delta_{12} = F_1 + F_2 - 2(C_0 + C_{12})$$

which mean that the eight components of Δ_{12} are exactly the eight components of the fibers F_1 and F_2 that appear with multiplicity one. Hence τ^*F_1 and τ^*F_2 are just smooth elliptic curves. However the six fibers F_3, \dots, F_8 split under the cover and define twelve I_2 fibers of the elliptic fibration on $\text{Kum}(B_{12})$ defined by τ^*F . Again a computation of Euler characteristics shows that these twelve I_2 fibers are the only singular fibers of the linear system $|\tau^*F|$. Also the sections C_{13}, C_{14}, C_{15} and C_{16} of $|F|$ pull back to sections of τ^*F , which is therefore a Weierstrass elliptic fibration. \square

We now proceed to the realization of the surface $\text{Kum}(B_{12})$ as a double cover of a weak del Pezzo surface. We decompose the sextic \mathcal{S} (see figure 1) into the quartic $Q = l_3 + l_4 + l_5 + l_6$ and the conic $C = l_1 + l_2$.

Theorem 5.2. *The rational double cover associated to Δ_{12} , $\text{Kum}(B_{12}) \dashrightarrow Y$ decomposes as*

$$\begin{array}{ccc} \text{Kum}(B_{12}) & \xrightarrow{\varphi} & T \\ \downarrow \tau & & \downarrow \zeta \\ Y & \xrightarrow{\phi} & \mathbb{P}^2 \end{array}$$

where ϕ is the canonical resolution of the double cover of \mathbb{P}^2 branched along \mathcal{S} . The maps ζ and φ are the canonical resolutions of the double covers branched along Q and $\zeta^*(C)$ respectively.

Proof. Let $T_0 \rightarrow \mathbb{P}^2$ be the double cover of \mathbb{P}^2 ramified over the reducible quartic Q . Its canonical resolution induces the diagram

$$\begin{array}{ccc} T & \xrightarrow{\quad} & T_0 \\ \downarrow & \searrow \zeta & \downarrow \\ \tilde{\mathbb{P}}^2 & \xrightarrow{\quad} & \mathbb{P}^2 \end{array}$$

where $\tilde{\mathbb{P}}^2 \rightarrow \mathbb{P}^2$ is the the blowup of \mathbb{P}^2 at the six singular points of Q . The surface T is a non-minimal rational surface containing six disjoint smooth rational curves. Indeed by Hurwitz formula, the canonical divisor of T is given by

$$K_T = \zeta^*(K_{\mathbb{P}^2} + \frac{1}{2}(l_3 + l_4 + l_5 + l_6)) = -\zeta^*(H)$$

where H is a hyperplane section. Thus $K_T^2 = 2, H^2 = 2$ and $P_2(T) = 0$. Denote by \tilde{Q} the proper transform of Q in T . Using the additivity of the topological Euler characteristic

and the Noether formula, we have that

$$e(T) = e(T - \tilde{Q}) + e(\tilde{Q}) = 10 \Rightarrow \mathcal{X}(\mathcal{O}_T) = 1 \Rightarrow q(T) = 0$$

By Castelnuovo's rationality criterion, T is a rational surface. In fact, we show that T is a weak del Pezzo surface of degree two, i.e. the blow up of \mathbb{P}^2 at seven points with nef canonical divisor. Indeed we successively blown down the preimages in T of the four lines l_3, l_4, l_5 and l_6 as well as the preimages in T of the three "diagonals" of the complete quadrangle formed by l_3, l_4, l_5, l_6 . The surface obtained after these seven blow down is a projective plane.

Consider the following curves of T

$$\zeta^*(C) = \zeta^*(l_1 + l_2) = E_1 + E_2, \text{ where } E_1 \text{ and } E_2 \text{ are smooth elliptic curves}$$

and

$$\zeta^*(W) = W_1 + W_2 \text{ where } W_1 \text{ and } W_2 \text{ are smooth rational curves}$$

with the following intersection properties:

$$E_i^2 = 2, \quad W_i^2 = 0, \quad E_1 \cdot E_2 = 2, \quad W_1 \cdot W_2 = 4, \quad W_i \cdot E_j = 2 \quad \text{for } i \neq j.$$

(recall that W is the plane conic tangent to the six lines l_1, \dots, l_6). The linear system $|E_1|$ defines an elliptic fibration on T with six singular fibers of type I_2 . Take the double cover branched along the two fibers $E_1 + E_2 \in 2\text{Pic}(T)$. It induces the canonical resolution commutative diagram:

$$\begin{array}{ccc} X & \longrightarrow & X_0 \\ & \searrow \varphi & \downarrow \\ \tilde{T} & \longrightarrow & T \end{array}$$

where $\tilde{T} \rightarrow T$ is the blowup of T at the two singular points of $E_1 + E_2$.

Claim: X is a Kummer surface.

Proof of the Claim: Clearly $\mathcal{K}_X = \varphi^*(\zeta^*(-H) + \frac{1}{2}(E_1 + E_2)) = \mathcal{O}_X$.

1) The pullback by φ of the six exceptional curves on T define twelve disjoint smooth rational curves on X .

2) The two exceptional curves of X give two more rational curves disjoint from 1).

3) Let $\varphi^*(W_1) = W'_1 + W''_1$ and $\varphi^*(W_2) = W'_2 + W''_2$ and let σ be the lift on X of the covering involution of ζ , then $\sigma(W'_1) = W'_2$ or $\sigma(W'_1) = W''_2$. Without loss of generality, we can assume that $\sigma(W'_1) = W'_2$ and hence get the following intersection numbers

$$W_i'^2 = W_i''^2 = -2, \quad W'_i \cdot W''_i = 2 \quad \text{for } i = 1, 2$$

$$W'_1 \cdot W'_2 = W''_1 \cdot W''_2 = 4 \quad \text{and} \quad W'_1 \cdot W''_2 = W''_1 \cdot W'_2 = 0.$$

One easily checks that W'_1 and W''_2 do not intersect the fourteen curves from 1) and 2). In particular, the $K3$ surface X contains sixteen disjoint smooth rational curves. Consequently X is a Kummer surface. Moreover, the surface X contains an elliptic fibration with twelve I_2 fibers. It also admits two non symplectic involutions θ and σ where θ is

the covering involution of the map φ and σ is the lift of the covering involution of ζ on T encountered earlier. The composition $\iota = \varphi \circ \sigma$ defines a symplectic involution on X whose quotient is a $K3$ surface admitting an elliptic fibration with singular fibers identical to the one defined by F on Y in the proposition 5.1.

In fact, we can now recover sixteen disjoint rational curves on the quotient and conclude that it is our original general Kummer surface Y and that $X \simeq \text{Kum}(B_{12})$.

□

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SECTION DE MATHÉMATIQUES, UNIVERSITÉ DE GENÈVE, 2-4 RUE DU LIÈVRE, CASE POSTALE 64, 1211 GENÈVE 4, SUISSE.,

E-mail address: Afsaneh.Mehran@math.unige.ch